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DEGREES OF FINITE-TO-ONE FACTOR MAPS

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ABSTRACT
If $f: \Sigma_A \to \Sigma_A$ is an endomorphism of an irreducible shift of finite type, and p is a rational prime which divides the degree of f, then p divides every non-leading coefficient of χ_A. We also give constraints on the possible degrees of finite-to-one factor maps between shifts of finite type.

0. Introduction

It is a curious fact that if Σ_A and Σ_B are irreducible shifts of finite type of equal entropy, and $f: \Sigma_A \to \Sigma_B$ is a factor map, then there is a positive integer d such that every point in Σ_B with dense forward and backward orbit, under the shift, has exactly d preimages. d is called the *degree* of f. Previous authors have shown that the degree of a factor map is not arbitrary. L. R. Welch showed that if f is an endomorphism of the full shift on n symbols, then the degree of f divides a power of n ([H, 14.9]). Next, M. Boyle proved that if S is an irreducible sofic system of entropy $\log \lambda$, and $f: S \to S$ is an endomorphism, then the degree of f is a unit in $\mathbf{Z}[1/\lambda]$ ([B1, Corollary 2.3]). See also [A] for interesting recent work on degrees.

In this paper, we prove that for an irreducible shift of finite type Σ_A, the degree of an endomorphism must be a unit in $\mathbf{Z}[1/\gamma]$ for *any* non-zero eigenvalue γ. This answers a question of Boyle in [B1]. We prove the following results.

THEOREM A. (See Theorem 2.7.) *Let Σ_A be an irreducible shift of finite type and γ a non-zero eigenvalue for A. Then there exists a finite set E of positive integers such that if $f: \Sigma_A \to \Sigma_B$ is a finite-to-one factor map*, $\deg(f) = d$, *and* $\operatorname{mult}(A,\gamma) = \operatorname{mult}(B,\gamma)$, *then $d = eu$, where $e \in E$ and u is a unit in $\mathbf{Z}[1/\gamma]$.* ($\operatorname{Mult}(A,\gamma)$ *is the algebraic multiplicity of γ in χ_A.*)

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COROLLARY B. (See Corollary 2.8.) *Let* Σ_A *be an irreducible shift of finite type and let* $f: \Sigma_A \to \Sigma_A$ *be an endomorphism,* $\deg(f) = d$. *Let* γ *be a non-zero eigenvalue for* A. *Then* d *is a unit in* $\mathbf{Z}[1/\gamma]$.

COROLLARY C. (See Corollary 2.11.) *Let* Σ_A *be an irreducible shift of finite type, and let* $f: \Sigma_A \to \Sigma_A$ *be an endomorphism,* $\deg(f) = d$. *If* p *is a rational prime dividing* d, *then* p *divides every non-leading coefficient of* χ_A.

In [T] we proved these results in the special case that f is constant-to-one (i.e. every point has exactly d inverse images). Theorem A and Corollaries B and C are generalizations of [B1, Theorem 2.2 and Corollary 2.4].

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1. Background

We assume that the reader has some familiarity with shifts of finite type and symbolic dynamics. For a more thorough introduction to the subject, see [AM] or [PT].

Let A be an $n \times n$, non-negative, integral matrix, and let $G(A)$ be the directed graph on n vertices, with A_{ij} edges from vertex i to vertex j. Let $\mathcal{S}_A = \{\text{vertices of } G(A)\}$. Elements of $\mathcal{S}_A$ are called *states*. Let $\mathcal{E}_A = \{\text{edges of } G(A)\}$. We define the *shift of finite type*, $\Sigma_A = \{x \in \mathcal{E}_A^{\mathbf{Z}} : x_{i+1} \text{ follows } x_i \text{ for all } i\}$, where x_{i+1} follows x_i if the terminal vertex of x_i is the initial vertex of x_{i+1}.

An A-*word* is a sequence $w = x_1 x_2 \cdots x_n \in \mathcal{E}_A^n$ such that x_{i+1} follows x_i, $1 \le i < n$. If a is the initial vertex of x_1, and a' is the terminal vertex of x_n, we say that w *begins* with x_1 (or begins with a) and w *ends* with x_n (or ends with a').

A is *irreducible* if, for each i, j, there exists $n = n(i,j)$ such that $A^n_{ij} > 0$. Σ_A is irreducible if A is. A *factor map* $f: \Sigma_A \to \Sigma_B$ is a continuous, surjective map which commutes with the shift. f is *finite-to-one* if there exists $N \in \mathbf{Z}^+$ such that $\# f^{-1}(y) \le N$ for all $y \in \Sigma_B$. By [CP1, Cor. 4.5], this is equivalent to $h(\Sigma_A) = h(\Sigma_B)$.

Our main tools in this paper will be an A-invariant subspace V of $\mathbf{C}^n$ and an A^T-invariant subspace W. The details of this construction are given in [KMT, section 4]. We review them here.

DEFINITION 1.1. Let $f: \Sigma_A \to \Sigma_B$ be a finite-to-one, one-block factor map, Σ_A irreducible. The *degree* of f (denoted $\deg(f)$) is defined by

$$\deg(f) = \inf_{\substack{y_1\cdots y_n \\ \text{a } B\text{-word}}} \inf_{1\le i\le n} \#\{a \in \mathcal{E}_A\colon \text{there exists an } A\text{-word } x_1\cdots x_n \text{ such that } x_i = a \text{ and } f(x_1\cdots x_n) = y_1\cdots y_n\}.$$

The degree of an arbitrary finite-to-one factor map is defined to be the degree of the one-block map obtained by passing to a higher block presentation (see [AM, p. 13]). It is easily verified that this definition is invariant under conjugacy.

A B-word for which the infimum occurs is called a *magic word*. For the case $d = 1$, this was called a resolving block in [AM]. A magic word of length one is called a *magic symbol*. Up to topological conjugacy of the domain and range, any finite-to-one factor map can be assumed to have a magic symbol (see [KMT, Theorem 2.3]). If $\deg(f) = d$ and w is a magic word for f, it is not hard to show that every $y \in \Sigma_B$ has at least d preimages, and every y, in which w occurs infinitely often to the right and the left, has exactly d preimages (see [KMT, Theorem 2.6]). In particular, every bilaterally transitive point has exactly d preimages. This fact was originally proved by G. Hedlund for endomorphisms of full shifts [H, 14.9], and later extended to shifts of finite type by E. Coven and M. Paul [CP2, Theorem 6.5].

In what follows, we assume that $f\colon \Sigma_A \to \Sigma_B$ is a finite-to-one one-block map, Σ_A irreducible, $\deg(f) = d$ and s is a magic symbol for f. Assume A is $n \times n$ and B is $k \times k$.

Definition 1.2. Let

$$R_B = \{B\text{-words which end with } s\} \quad \text{and}$$

$$L_B = \{B\text{-words which begin with } s\}.$$

If $a \in \mathcal{S}_B$, let

$$R_B^a = \{B\text{-words } w \in R_B\colon w \text{ begins with } a\} \quad \text{and}$$

$$L_B^a = \{B\text{-words } w \in L_B\colon w \text{ ends with } a\}.$$

For each $w \in R_B$, and $\alpha \in f^{-1}(s)$, define a $1 \times n$ row vector $r^{(w,\alpha)}$ by

$$(r^{(w,\alpha)})_i = \begin{cases} 1 & \text{if there exists } u \in f^{-1}(w) \text{ beginning with } i \in \mathcal{S}_A \\ & \quad \text{and ending with } \alpha \in f^{-1}(s), \\ 0 & \text{otherwise.} \end{cases}$$

In [KMT], this was defined to be an $n \times 1$ column vector, so that several statements in [KMT] need to be transposed to agree with our notation. (T will denote the transpose of a matrix or vector.)

Similarly, if $w' \in L_B$, let $l^{(w',\alpha)}$ be the $1 \times n$ row vector defined by

$$(l^{(w',\alpha)})_i = \begin{cases} 1 & \text{if there exists } v \in f^{-1}(w') \text{ ending with } i \in \mathcal{S}_A \\ & \qquad \text{and beginning with } \alpha \in f^{-1}(s), \\ 0 & \text{otherwise.} \end{cases}$$

Let $r^w = \sum_{\alpha \in f^{-1}(s)} r^{(w,\alpha)}$ and $l^{w'} = \sum_{\alpha \in f^{-1}(s)} l^{(w',\alpha)}$.

Lemma 1.3. *If $v \in R_B$, $w \in L_B$, $\alpha \in f^{-1}(s)$, then*

(i) $$l^{(w,\alpha)} r^{v^T} = \begin{cases} 1 & \textit{if } v \in R_B^a \textit{ and } w \in L_B^a \textit{ for some } a \in \mathcal{S}_B, \\ 0 & \textit{otherwise}; \end{cases}$$

$$l^w r^{(v,\alpha)^T} = \begin{cases} 1 & \textit{if } v \in R_B^a \textit{ and } w \in L_B^a \textit{ for some } a \in \mathcal{S}_B, \\ 0 & \textit{otherwise}; \end{cases}$$

(ii) $$l^w r^{v^T} = \begin{cases} d & \textit{if } w \in L_B^a \textit{ and } v \in R_B^a \textit{ for some } a \in \mathcal{S}_B, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof of (i) is contained in the first part of the proof of [KMT, Lemma 4.5]. To prove (ii), simply observe that if $f^{-1}(s) = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$, then $l^w r^{v^T} = \sum_{i=1}^d l^{(w,\alpha_i)} r^{v^T} = d$, by (i). □

Definition 1.4. Let

$$V = \text{complex span } \{l^w : w \in L_B\} \quad \text{and} \quad W = \text{complex span } \{r^v : v \in R_B\}.$$

Lemma 1.5. *V is A-invariant and W is A^T-invariant.*

Proof. The first statement follows from [KMT, Lemma 4.3]. The second is proved similarly, observing that if $v \in R_B$, then $r^v A^T = \sum_{a \in \mathcal{P}(v)} r^{av}$, where $\mathcal{P}(v) = \{a \in \mathcal{E}_B : av \text{ is allowed}\}$.

Lemma 1.6. (See [KMT, Lemma 4.5].) *There exist surjective linear maps $\phi : V \to \mathbf{C}^k$ and $\theta : W \to \mathbf{C}^k$ such that, for $x \in V$, $y \in W$, $xA\phi = x\phi B$ and $yA^T\theta = y\theta B^T$.*

Proof. The proof of [KMT, Lemma 4.5] shows that there is a matrix M such that $l^{(w,\alpha)}MB = l^{(w,\alpha)}AM$ for all $w \in L_B$. Summing over $\alpha \in f^{-1}(s)$ yields $l^w MB = l^w AM$. Observing that $l^w M = de_a$, if $w \in L_B^a$ (where e_a is a standard basis vector), proves the first statement. The second is proved similarly: θ is represented by an $n \times k$ matrix N, in which column a is $l^{w_a^T}$, where $w_a \in L_B^a$. □

2. Main results

DEFINITION 2.1. Let $\mathcal{R}$ be a subring of $\mathbf{C}$ containing 1. An $\mathcal{R}$-*vector* is a vector $v \in \mathbf{C}^n$ all of whose components lie in $\mathcal{R}$.

NOTATION. Let A be an $n \times n$ matrix, and γ an eigenvalue for A. Let

$$\mathcal{G}(A,\gamma) = \{v \in \mathbf{C}_n : v(A - \gamma I)^i = 0 \text{ for some } i \in \mathbf{Z}^+\}$$

denote the generalized eigenspace for A.

Let $\mathrm{mult}(A,\gamma)$ denote the algebraic multiplicity of γ in χ_A. Clearly, $\mathrm{mult}(A,\gamma) = \dim\mathcal{G}(A,\gamma) = \dim\mathcal{G}(A^T,\gamma)$.

LEMMA 2.2. *Let A be an $n \times n$ integral matrix, and γ a non-zero eigenvalue for A. Let $l \in \mathcal{G}(A,\gamma)$ and $r \in \mathcal{G}(A^T,\gamma)$ and suppose l and r are $\mathbf{Z}[1/\gamma]$-vectors. If A is shift equivalent over $\mathbf{Z}$ to $\bar{A}$, then there exist $\bar{l} \in \mathcal{G}(\bar{A},\gamma)$ and $\bar{r} \in \mathcal{G}(\bar{A}^T,\gamma)$, both of which are $\mathbf{Z}[1/\gamma]$-vectors, such that $\bar{l}\bar{r}^T = lr^T$.*

PROOF. Since A is shift equivalent over $\mathbf{Z}$ to $\bar{A}$, we have $AR = R\bar{A}$, $SA = \bar{A}S$, $RS = A^m$, $SR = \bar{A}^m$, for integral matrices R and S and a positive integer m. (See [W1] for definitions.) We claim that A acts invertibly on the module $M = \mathcal{G}(A,\gamma) \cap (\mathbf{Z}[1/\gamma])^n$. To see this, observe that since $l(A - \gamma I)^i = 0$, for some i, we may solve for l to obtain $l = lp(A)$, where $p(x)$ is a polynomial over $\mathbf{Z}[1/\gamma]$ with constant term 0. Therefore, $l = lq(A)A$, where $lq(A) \in M$, so $A|_M$ is surjective. Since A is clearly one-to-one on M, it is invertible. It follows that there exists $l' \in M$ such that $l'A^m = l$. Now, let $\bar{l} = l'R$ and $\bar{r} = rS^T$. Clearly $\bar{l}$ and $\bar{r}$ are $\mathbf{Z}[1/\gamma]$-vectors. It follows from the shift equivalence equations that $\bar{l} \in \mathcal{G}(\bar{A},\gamma)$ and $\bar{r} \in \mathcal{G}(\bar{A}^T,\gamma)$, and we have $\bar{l}\bar{r}^T = l'RSr^T = l'A^m r^T = lr^T$. □

LEMMA 2.3. *Let $f: \Sigma_A \to \Sigma_B$ be a finite-to-one, one-block factor map, Σ_A irreducible, with magic symbol s, and let V and W be the subspaces in Definition* 1.4. *Let γ be a non-zero eigenvalue for A. If* $\mathrm{mult}(A,\gamma) = \mathrm{mult}(B,\gamma)$, *then* $\mathcal{G}(A,\gamma) \subseteq V$ *and* $\mathcal{G}(A^T,\gamma) \subseteq W$.

PROOF. Assume that B is $k \times k$. By Lemma 1.6, there exists a surjective linear map $\phi : V \to \mathbf{C}^k$ such that for all $v \in V$, $vA\phi = v\phi B$. By duality, it follows that V contains $\mathrm{mult}(B,\gamma)$ linearly independent vectors in $\mathcal{G}(A,\gamma)$. Since $\dim\mathcal{G}(A,\gamma) = \mathrm{mult}(A,\gamma) = \mathrm{mult}(B,\gamma)$, these vectors must span $\mathcal{G}(A,\gamma)$, and so $\mathcal{G}(A,\gamma) \subseteq V$.

A similar argument shows that $\mathcal{G}(A^T,\gamma) \subseteq W$. □

In Lemmas 2.4 and 2.5 we assume that $f\colon \Sigma_A \to \Sigma_B$ is a finite-to-one one-block factor map with magic symbol s, $\deg(f) = d$, that Σ_A is irreducible, and that R is a subring of $\mathbf{C}$ containing 1.

LEMMA 2.4. *Let $\sum c_i l^{w_i}$ and $\sum d_i r^{v_i}$ be $\mathcal{R}$-vectors, $c_i, d_i \in \mathbf{C}$. Then for each $a \in \mathcal{S}_B$,*

$$\sum_{i:\, w_i \in L_B^a} c_i \in \mathcal{R} \quad \textit{and} \quad \sum_{i:\, v_i \in R_B^a} d_i \in \mathcal{R}.$$

PROOF. Given $a \in \mathcal{S}_B$, choose $v \in R_B^a$ and $\alpha \in f^{-1}(s)$. Then

$$\left(\sum_i c_i l^{w_i}\right) r^{(v,\alpha)^T} = \sum_{i:\, w_i \in L_B^a} c_i$$

by Lemma 1.3(i). Since the left side is the product of two $\mathcal{R}$-vectors, it lies in $\mathcal{R}$. A similar argument (using $l^{(w,\alpha)}$) proves the second assertion. □

LEMMA 2.5. *If $l = \sum c_i l^{w_i}$ and $r = \sum d_j r^{v_j}$ are $\mathcal{R}$-vectors, $c_i, d_j \in \mathbf{C}$, then d divides lr^T in $\mathcal{R}$.*

PROOF.

$$\begin{aligned} lr^T &= \left(\sum_i c_i l^{w_i}\right)\left(\sum_j d_j r^{v_j}\right)^T \\ &= \sum_{i,j} c_i d_j l^{w_i} r^{v_j^T} \\ &= \sum_{a \in \mathcal{S}_B} \sum_{\substack{i,j:\, w_i \in L_B^a \\ v_j \in R_B^a}} c_i d_j l^{w_i} r^{v_j^T} \\ &= \sum_{a \in \mathcal{S}_B} \left[\sum_{i:\, w_i \in L_B^a} c_i \sum_{j:\, v_j \in R_B^a} d_j\right] d. \end{aligned}$$

The last two steps follow from Lemma 1.3(ii). Since both sums inside the brackets are in $\mathcal{R}$, by Lemma 2.4, the entire sum is divisible by d in $\mathcal{R}$. □

LEMMA 2.6. *Let $f\colon \Sigma_A \to \Sigma_B$ be a finite-to-one factor map*, $\deg(f) = d$. *Let γ be a non-zero eigenvalue for A, and let $l \in \mathcal{G}(A,\gamma)$ and $r \in \mathcal{G}(A^T,\gamma)$. Assume that l and r are $\mathbf{Z}[1/\gamma]$-vectors and that* $\operatorname{mult}(A,\gamma) = \operatorname{mult}(B,\gamma)$. *Then d divides lr^T in $\mathbf{Z}[1/\gamma]$.*

PROOF. By [KMT, Theorem 2.3], there exists $\Sigma_{\bar{A}}$ topologically conjugate to Σ_A, a positive integer N, and a one-block map $\bar{f}\colon \Sigma_{\bar{A}} \to \Sigma_{B^{[N]}}$, so that $\bar{f}$

has a magic symbol s. By [W2, Corollary 4.8], $\mathrm{mult}(\bar{A},\gamma) = \mathrm{mult}(A,\gamma)$ and $\mathrm{mult}(B^{[N]},\gamma) = \mathrm{mult}(B,\gamma)$, so $\mathrm{mult}(\bar{A},\gamma) = \mathrm{mult}(B^{[N]},\gamma)$. Since Σ_A is topologically conjugate to $\Sigma_{\bar{A}}$, A and $\bar{A}$ are shift equivalent (see [W1]). By Lemma 2.2, there exists $\bar{l} \in \mathcal{G}(\bar{A},\gamma)$ and $\bar{r} \in \mathcal{G}(\bar{A}^T,\gamma)$, both of which are $\mathbf{Z}[1/\gamma]$-vectors, such that $\bar{l}\bar{r}^T = lr^T$. Let V and W be the subspaces of Definition 1.4. By Lemma 2.3, $\mathcal{G}(\bar{A},\gamma) \subseteq V$ and $\mathcal{G}(\bar{A}^T,\gamma) \subseteq W$, so that $\bar{l} \in V$ and $\bar{r} \in W$. By Lemma 2.5, with $\mathcal{R} = \mathbf{Z}[1/\gamma]$, d divides $\bar{l}\bar{r}^T$, and therefore lr^T, in $\mathbf{Z}[1/\gamma]$. □

We can now state our main results.

THEOREM 2.7. *Let Σ_A be an irreducible shift of finite type, and γ a non-zero eigenvalue for A. Then there exists a finite set E of positive integers such that if $f: \Sigma_A \to \Sigma_B$ is a finite-to-one factor map,* $\deg(f) = d$, *and γ is an eigenvalue for B, with* $\mathrm{mult}(A,\gamma) = \mathrm{mult}(B,\gamma)$, *then $d = eu$, where $e \in E$ and u is a unit in* $\mathbf{Z}[1/\gamma]$.

COROLLARY 2.8. *Let Σ_A be an irreducible shift of finite type and let f: $\Sigma_A \to \Sigma_A$ be an endomorphism,* $\deg(f) = d$. *Let γ be a non-zero eigenvalue for A. Then d is a unit in* $\mathbf{Z}[1/\gamma]$.

PROOF OF 2.7. First, choose $l \in \mathcal{G}(A,\gamma)$, and $r \in \mathcal{G}(A^T,\gamma)$, both of which are $\mathbf{Z}[1/\gamma]$-vectors, and such that $lr^T \neq 0$. To see that we can do this, first choose $\bar{l} \in \mathcal{G}(A,\gamma)$, which is a $\mathbf{Q}[\gamma]$-vector. Since for any eigenvalue $\alpha \neq \gamma$, and $v \in \mathcal{G}(A^T,\alpha)$, $\bar{l}v^T = 0$, and since the generalized eigenvectors span $\mathbf{C}^n$ (where A is $n \times n$), there exists $\bar{r} \in \mathcal{G}(A^T,\gamma)$ such that $\bar{l}\bar{r}^T \neq 0$, and we may assume $\bar{r}$ is a $\mathbf{Q}[\gamma]$-vector. Now, multiply $\bar{l}$ and $\bar{r}$ by a sufficiently large integer to obtain generalized eigenvectors l and r, which are $\mathbf{Z}[\gamma]$-vectors, and therefore $\mathbf{Z}[1/\gamma]$-vectors (since $\mathbf{Z}[\gamma] \subseteq \mathbf{Z}[1/\gamma]$).

The remainder of the proof now continues essentially as in the proof of [B1, Theorem 2.2], although we are applying that argument to all non-zero eigenvalues, rather than just the Perron value.

Since $1/lr^T \in \mathbf{Q}[\gamma]$, there exists a positive integer m such that $m/lr^T \in \mathbf{Z}[\gamma] \subseteq \mathbf{Z}[1/\gamma]$. Let E be the set of positive integers dividing m. If $f: \Sigma_A \to \Sigma_B$ is finite-to-one and $\mathrm{mult}(A,\gamma) = \mathrm{mult}(B,\gamma)$, then $lr^T/d \in \mathbf{Z}[1/\gamma]$ by Lemma 2.6. So

$$\frac{m}{d} = \frac{m}{lr^T}\frac{lr^T}{d} \in \mathbf{Z}\left[\frac{1}{\gamma}\right].$$

Write $d = eu$, where $e = gcd(m,d) \in E$. Let $k = m/e$. Then

$$\frac{k}{u} = \frac{ke}{ue} = \frac{m}{d} \in \mathbf{Z}\left[\frac{1}{\gamma}\right],$$

with k and u relatively prime. There exist integers x and y such that $kx + uy = 1$, so

$$\frac{k}{u}x + y = \frac{1}{u}$$

and therefore $1/u \in \mathbf{Z}[1/\gamma]$. □

PROOF OF 2.8. This argument appears in [B1, Corollary 2.4]; again, we are applying the argument to all non-zero eigenvalues rather than just the Perron value.

Given Σ_A, let E be the finite set guaranteed by Theorem 2.7. If f is an endomorphism of Σ_A, $\deg(f) = d$, then $d = eu$, where $e \in F$ and u is a unit in $\mathbf{Z}[1/\gamma]$. Now $\deg(f^k) = e^k u^k$, which forces some power of e, and therefore e itself, to be a unit in $\mathbf{Z}[1/\gamma]$. □

EXAMPLE 2.9. Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

Since $\gamma = -1$ is an eigenvalue for A, if $f: \Sigma_A \to \Sigma_A$ is an endomorphism, $\deg(f) = d$, then by Corollary 2.9, d is a unit in $\mathbf{Z}[1/\gamma] = \mathbf{Z}$. Therefore $d = 1$.

The rational units of $\mathbf{Z}[1/\gamma]$ can be computed from the following proposition.

PROPOSITION 2.10. *Let γ be an algebraic integer with minimal polynomial $r(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, and suppose $d \in \mathbf{Z}$. Then the following conditions are equivalent.*

(1) *d is a unit in $\mathbf{Z}[1/\gamma]$.*

(2) *If p is a rational prime dividing d, then p divides a_i, $0 \le i \le n-1$.*

PROOF. See [B1, Proposition 2.3].

COROLLARY 2.11. *Suppose $f: \Sigma_A \to \Sigma_A$ is an endomorphism, Σ_A irreducible, $\deg(f) = d$. If p is a rational prime dividing d, then p divides every non-leading coefficient of χ_A.*

PROOF. By Gauss' lemma, we can factor χ_A into a product of irreducible, monic polynomials with integer coefficients, each of which is the minimal polynomial for some of the eigenvalues of A. By Corollary 2.8 and Proposition 2.10, p divides the non-leading coefficients of each of these factors. (This holds trivially for the factors, if any, corresponding to eigenvalue 0.) It follows that p divides the non-leading coefficients of χ_A. □

We do not know whether the condition in Theorem 2.7 that $\mathrm{mult}(A,\lambda) = \mathrm{mult}(B,\lambda)$ can be weakened in general. It can in special cases. For example, if there exists an actual eigenvector l for A (not generalized), and an eigenvector r for A^T such that $lr^T \neq 0$, then it is enough to assume that the dimensions of the γ-eigenspaces for A and B are equal. This is so because the equality of dimensions forces $l \in V$ and $r \in W$, and the proof of Theorem 2.7 continues as before. This really is a weaker condition, because if Σ_A factors onto Σ_B, then $\mathrm{mult}(A,\gamma) = \mathrm{mult}(B,\gamma)$ implies that the dimensions of the γ-eigenspaces for A and B are equal (since the linear map ϕ of Lemma 1.6, restricted to $\mathcal{G}(A,\gamma)$, is an isomorphism). However, for some matrices we may have $lr^T = 0$ for all eigenvectors l and r.

If A has a unique Jordan block of largest dimension, then the condition $\mathrm{mult}(A,\gamma) = \mathrm{mult}(B,\gamma)$ can be replaced by the condition that B have a Jordan block of the same size. This is so because the Jordan form away from 0 of B must sit inside the Jordan forms away from 0 of $A|_V$ and $A^T|_W$ (see [Ki] or use Lemma 1.6). So choosing l and r from the A-invariant subspace and the A^T-invariant subspace, respectively, corresponding to the largest Jordan block (with $lr^T \neq 0$), forces l to lie in V and r to lie in W.

M. Boyle has observed that there is a partial converse to Corollary 2.11.

THEOREM 2.12 (Boyle, [B2]). *Let A be a non-negative, integral matrix, whose minimal polynomial has degree m, and suppose that d is a positive integer which is a unit in $\mathbf{Z}[1/\gamma]$, for every non-zero eigenvalue γ for A. If $n \geq m$, then there exists a constant-to-one endomorphism of Σ_{A^n} of degree d.*

PROOF. See [T, p. 188].

An example due to Boyle and Handelman shows that Theorem 2.12 cannot be improved to obtain a constant-to-one endomorphism of Σ_A of degree d (see [BH, example 6.2]). However, the question remains open whether, under the hypotheses of Theorem 2.12, there exists an endomorphism of Σ_A (not necessarily constant-to-one) of degree d.

Corollary 2.8 generalizes to the case of sofic systems which are almost of finite type (called AFT sofic systems—see [BKM] for definitions).

THEOREM 2.13. *Let S be an irreducible AFT sofic system, presented by the factor map $\pi : \Sigma_A \to S$, where Σ_A* is irreducible and *π is 1-1 on an open set. Let γ be a non-zero eigenvalue for A. If $f: S \to S$ is an endomorphism,* $\deg(f) = d$, *then d is a unit in $\mathbf{Z}[1/\gamma]$.*

PROOF. By [BKM, Theorem 9], the map $f\pi$ induces a factor map $g: \Sigma_A \to \Sigma_A$ such that $f\pi = \pi g$. So $\deg(g) = d$. Now apply Corollary 2.8. □

If S is not AFT, but f is right closing, constraints on the possible degrees of $f: \Sigma_A \to S$ are imposed by the non-zero eigenvalues of the future cover of S, since f must factor through the future cover by a factor map of degree d (see [BKM, Prop. 4]).

In particular, if $f: \Sigma_2 \to S$ is right closing, then the future cover of S must be shift equivalent to Σ_2, by [BKM, Prop. 4] and [M, Theorem 8], so f induces an eventual endomorphism of Σ_2 (since Σ_2 is eventually conjugate to the future cover; see [BMT, Theorem 2.16]). Therefore the degree of f is a power of 2. Can we remove the assumption that f is right closing and obtain the same conclusion?

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